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# On unknotting operations of rotation type

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ABSTRACT

An unknotting operation is a local move on a knot diagram such that any knot diagram can be transformed into a diagram of the unknot by a finite sequence of the operations and Reidemeister moves. In this paper, we introduce a new local move $H(T)$ on a knot diagram which is obtained by the rotation of a tangle diagram $T$ and study their properties. As an application, we prove that the $H(T)$-move is an unknotting operation for any descending tangle diagram $T$.

*Keywords*: Unknotting operation; descending tangle diagram; chord-tangle diagram.

Mathematics Subject Classification 2010: 57M25, 57M27

## 1. Introduction

A local move on a knot diagram is called an *unknotting operation* if any knot diagram can be transformed into a diagram of the unknot by a finite sequence of the operations and Reidemeister moves. A well-known unknotting operation is the *crossing change* which changes the overcrossing and the undercrossing at a crossing of the diagram. Besides the crossing change, there are many unknotting operations, see Fig. 1, for example, the $\sharp$-move, the $\triangle$-move, the $n$-gon move, the $C_1$ move and the $C_2$ move and region crossing change, etc. [1, 6–8, 10]. In 1990, Hoste, Nakanishi and Taniyama [4] introduced the $H(n)$-move and proved that the $H(n)$-move is an unknotting operation for $n \geq 2$. In fact, the $H(2)$-move was first considered by Lickorish [5] as an unknotting operation obtained by adding a twisted band; see Fig. 2. In 1992, Ohyama [9] introduced the $n(1)$ and the $n(2)$-move and classified the $H(n)$-move and the $n(i)$-move as "unknotting operations of rotation type".

Let $K$ and $K'$ be knots in $S^3$ and $u$ an unknotting operation. The *u-Gordian distance* between $K$ and $K'$ is the minimum number of the unknotting operations

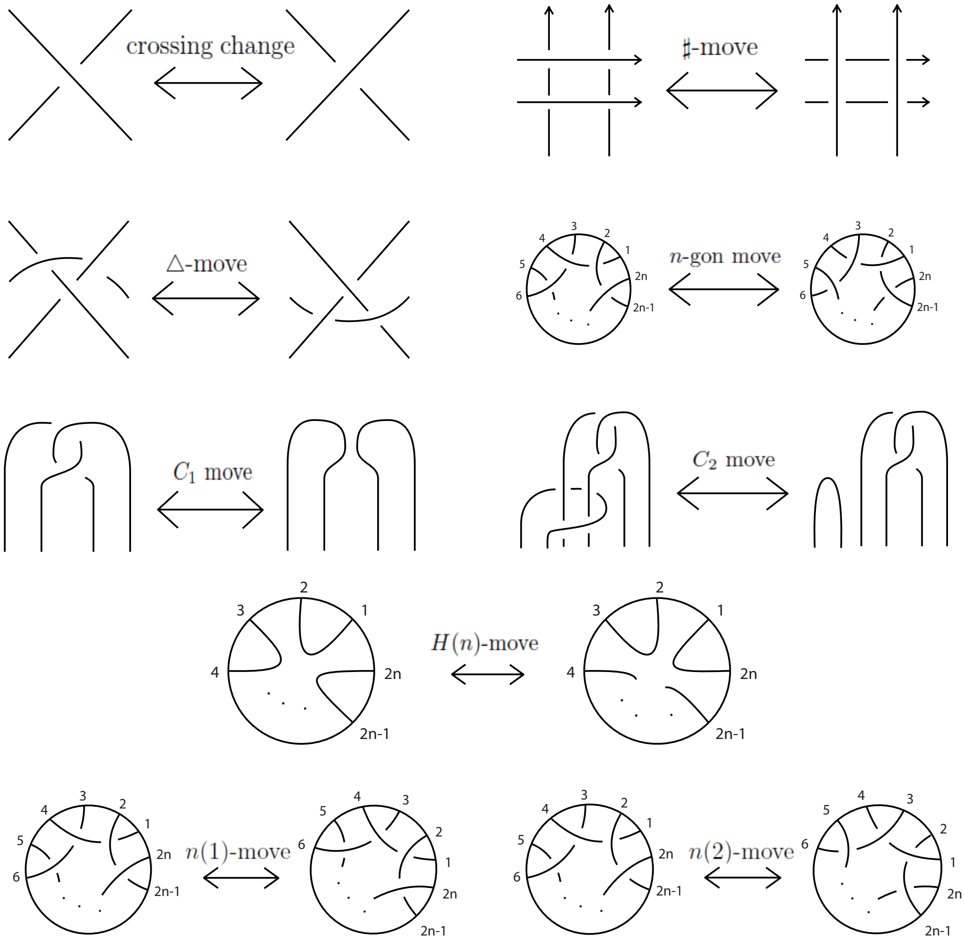


Fig. 1. Unknotting operations.

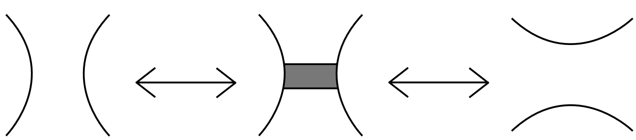

Fig. 2. $H(2)$-move.

needed to deform a diagram of $K$ into that of $K'$, where the minimum is taken over all diagrams of $K$ from which one can obtain a diagram of $K'$, and it is denoted by $d_u(K, K')$. In particular, the *u-unknotting number* of $K$ is the $u$-Gordian distance between $K$ and the unknot. It is known that $d_u$ is a metric on the set of all equivalence classes of knots in $S^3$. Many Gordian distances are defined and studied, for example, the usual Gordian distance $d(K, K')$ defined by the crossing change, $H(2)$-Gordian distance $d_2(K, K')$ defined by the $H(2)$-move and the $\triangle$-Gordian distance $d_\triangle(K, K')$ defined by the $\triangle$-move, etc.

In this paper, we introduce a new local move $H(T)$ on knot diagram which is obtained by the rotation of a tangle diagram $T$. The $H(T)$-move can be considered as "unknotting operations of rotation type". And as an application, we prove that the $H(T)$-move is an unknotting operation for any descending tangle diagram $T$.

## 2. Unknotting Operation Involving Tangle Diagram

An $n$-tangle diagram is a part of a link diagram with $2n$ free ends and it can be depicted as a disk in the plane such that the free ends of the diagram are on the boundary of the disk and the rest of the diagram are confined to the interior of the disk. Throughout this paper, we only consider tangle diagram without circle components.

An $n$-tangle diagram is said to be *descending* [2] if one can assign suitable height to each arc so that the crossing between arcs are compatible with the height. The first tangle diagram in Fig. 3 is descending. But the second tangle diagram is not descending because there are alternating cycles in it.

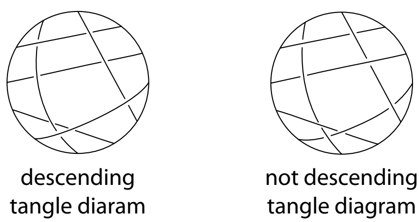


Fig. 3. Descending tangle diagram.

**Definition 2.1.** Let $T$ be an $n$-tangle diagram. An $H(T)$-move is a local move on a knot diagram rotating $T$ through $\frac{\pi}{n}$ as described in Fig. 4. We require an $H(T)$-move to preserve the number of component.

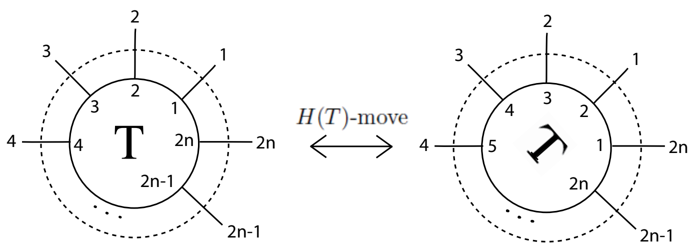


Fig. 4. $H(T)$-move.

**Example 2.2.** If $T_1, T_2, T_3$ are the diagrams in Fig. 5, respectively, then the $H(T_1)$-move is the $H(2)$-move, and the $H(T_2)$-move is the crossing change, and the $H(T_3)$-move is the $\triangle$-move.

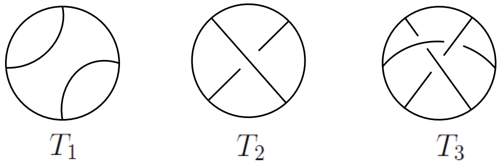


Fig. 5. $H(2)$ move, crossing change and $\Delta$-move.

None of the $\sharp$-move, the $n$-gon move and the $C_n$ move is a type of the $H(T)$-move.

**Theorem 2.1.** *Let $T$ be an $n$-tangle diagram. Let $T_\alpha$ denote the $(n+1)$-tangle diagram obtained from $T$ by adding a new arc $\alpha$ having either the top height or the bottom height. Then the $H(T)$-move can be realized by exactly one $H(T_\alpha)$-move.*

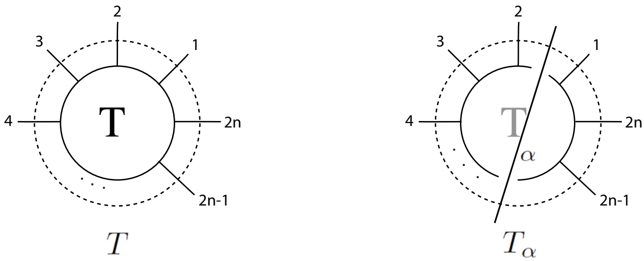


Fig. 6. $T_\alpha$.

**Proof.** Suppose that the tangle diagrams $T$ and $T_\alpha$ are of the form in Fig. 6. We may assume that the arc $\alpha$ has a top height in $T_\alpha$.

Let $\beta$ denote one of the arcs in $T$ which are adjacent to $\alpha$ in $T_\alpha$. Let $T'$ denote the new $n$-tangle diagram obtained from $T_\alpha$ by connecting $\alpha$ and $\beta$ as described at the middle of Fig. 7. Indeed, if $a$ and $b$ are the adjacent ends of $\alpha$ and $\beta$, respectively, and if $a'$ is the other end of $\alpha$, then connect $a'$ and $b$ by a path lying outside $T$ and over cross all arcs of $T_\alpha$. Note that $T'$ is ambient isotopic to the original tangle diagram $T$.

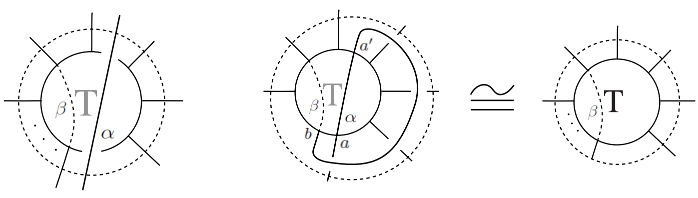


Fig. 7. Tangle diagram $T'$.

By applying the $H(T_\alpha)$-move to $T_\alpha$ in $T'$, we get the tangle diagram at the right-top in Fig. 8. Note that the resulting diagram is ambient isotopic to the diagram at the right-bottom in Fig. 8 which is the result of the $H(T)$-move to $T$. $\square$

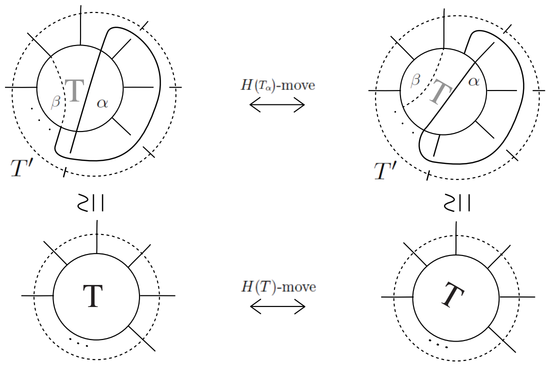


Fig. 8. $H(T_\alpha)$-move.

**Corollary 2.2.** *Let $T$ be an $n$-tangle diagram and $T_\alpha$ the tangle diagram defined in Theorem* 2.1. *If the $H(T)$-move is an unknotting operation, so does the $H(T_\alpha)$-move.*

For given unknotting operation $H(T)$-move, we denote the $H(T)$-Gordian distance between two knots $K$ and $K'$ by $d_{H(T)}(K, K')$.

**Corollary 2.3.** *Let $T$ be an $n$-tangle diagram such that the $H(T)$-move is an unknotting operation and $T_\alpha$ the tangle diagram defined in Theorem* 2.1. *Then*

$$d_{H(T)}(K, K') \geq d_{H(T_\alpha)}(K, K')$$

**Example 2.3.** Let $T, T'$ and $T''$ be the tangle diagrams in Fig. 9, respectively. By removing the top arc of each tangle diagram, one can see that the resulting tangle diagram moves are the $H(2)$-move, the crossing change and the $\triangle$-move, respectively. Hence the $H(T)$-move, the $H(T')$-move, and the $H(T'')$-move are unknotting operations. Hence one can see that for two knots $K$ and $K'$, $d_2(K, K') \geq d_{H(T)}(K, K')$, $d(K, K') \geq d_{H(T')}(K, K')$ and $d_\triangle(K, K') \geq d_{H(T'')}(K, K')$ by Corollary 2.3.

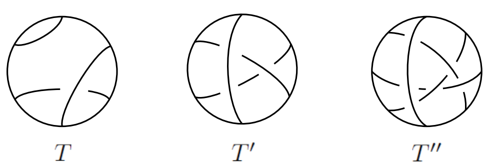


Fig. 9. Tangle diagrams with a top arc.

**Remark 2.4.** In Corollary 2.3, the equality does not hold in general. In [5], Lickorish showed that the figure-eight knot $(4_1)$ cannot be changed to the unknot by a single $H(2)$-move, and in fact, $d_2(4_1, O) = 2$. But $4_1$ can be changed to the unknot by applying one $H(T)$-move, where $T$ is given in Fig. 10. Hence $d_{H(T)}(4_1, O) = 1$.

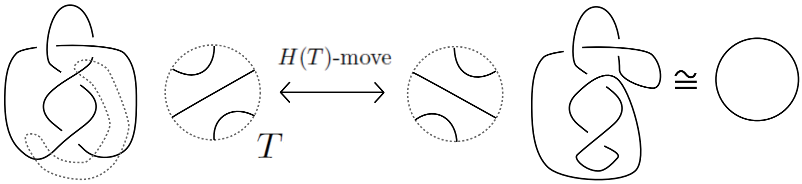


Fig. 10. The figure-eight knot.

**Corollary 2.5.** *The $H(T)$-move is an unknotting operation for any descending $n$-tangle diagram $(n \geq 2)$.*

**Proof.** Let $T$ be a descending $n$-tangle diagram. We use the induction on $n$. If $n = 2$, the only descending 2-tangle diagrams $T$ are $T_1$ and $T_2$ in Fig. 5, so that the $H(T)$-move is an unknotting operation for any descending 2-tangle diagram.

Assume that the $H(T)$-move is an unknotting operation for any descending $(n-1)$-tangle diagram $T$. Let $T$ is a descending $n$-tangle diagram. Let $T\backslash\alpha$ denote the $(n-1)$-tangle diagram obtained from $T$ by removing the top arc $\alpha$. By induction hypothesis, the $H(T\backslash\alpha)$-move is an unknotting operation. Since $T = (T\backslash\alpha)_\alpha$, by Corollary 2.2, the $H(T)$-move is an unknotting operation, too. □

**Corollary 2.6.** *For any descending $n$-tangle diagram $T$,*

$$d_{H(T)}(K, K') \leq \max\{d_2(K, K'), d(K, K')\}.$$

**Proof.** Since $T$ is descending, we can find the arc $\alpha$ which has the top height in $T$ and $\beta$ which has the second height in $T$. By removing extra arcs in sequence of low height, finally we get the 2-descending tangle diagram of the form either $T_1$ or $T_2$ in Fig. 5. Since the $H(T_1)$-move is the $H(2)$-move and the $H(T_2)$ move is the crossing change, either $d_{H(T)}(K, K') \leq d_2(K, K')$ or $d_{H(T)}(K, K') \leq d(K, K')$. □

A *chord diagram* [3] of order $n$ (or degree $n$) is an oriented circle with a distinguished set of $n$ disjoint pairs of distinct points, considered up to orientation preserving diffeomorphisms of the circle. The line which connects points of each pair is called a *chord*. In a chord diagram, it is assumed that the intersection number between two chords is 0 or 1.

**Definition 2.4.** An $n$-tangle diagram is called an *$n$-chord-tangle diagram* if its tangle shadow is one of chord diagrams of order $n$.

In Fig. 11, the tangle diagrams (a), (b) are 2-chord-tangle diagrams, but the tangle diagram in (c) is not a 2-chord-tangle diagram.

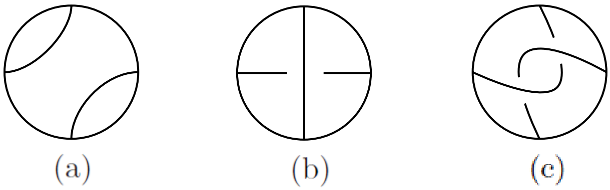


Fig. 11. Chord-tangle diagram.



Since the only two 2-chord-tangle diagrams are the tangle diagram (a) and the tangle diagram (b) in Fig. 11, the $H(T)$-move is an unknotting operation for any 2-chord-tangle diagram.

**Theorem 2.7.** *The $H(T)$-move is an unknotting operation for any 3-chord-tangle diagram $T$.*

**Proof.** A chord diagram of order 3 is one of the diagrams in Fig. 12.

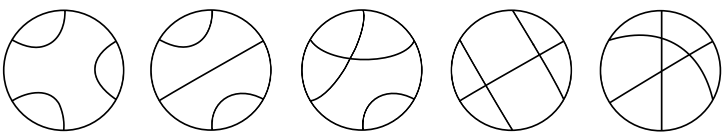

Fig. 12. 3-Chord diagrams.

All 3-chord-tangle diagrams are listed in Fig. 13.

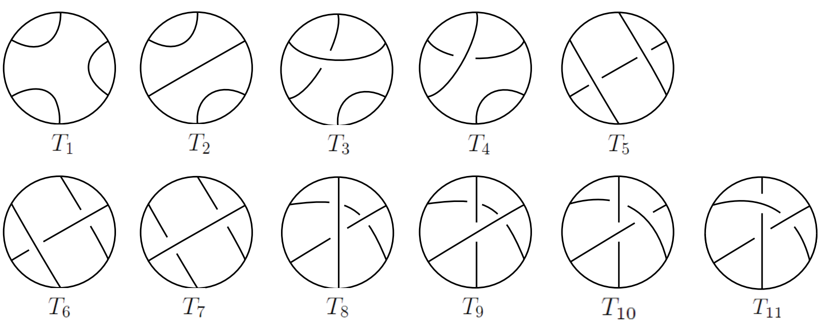


Fig. 13. 3-Chord-tangle diagrams.

For $T = T_i (i = 1, 2, \ldots, 9)$, there is the top arc. By removing the top arc, we get a 2-chord-tangle diagram. Since the $H(T)$-move is an unknotting operation for any 2-chord-tangle diagram $T$, the $H(T_i)$-move is an unknotting operation for $i = 1, 2, \ldots, 9$ by Corollary 2.2.

Notice that the $H(T_{10})$-move and the $H(T_{11})$-move are also unknotting operations because both of them are the $\triangle$-moves. □

A chord diagram of order 4 is one of the diagrams in Fig. 14.

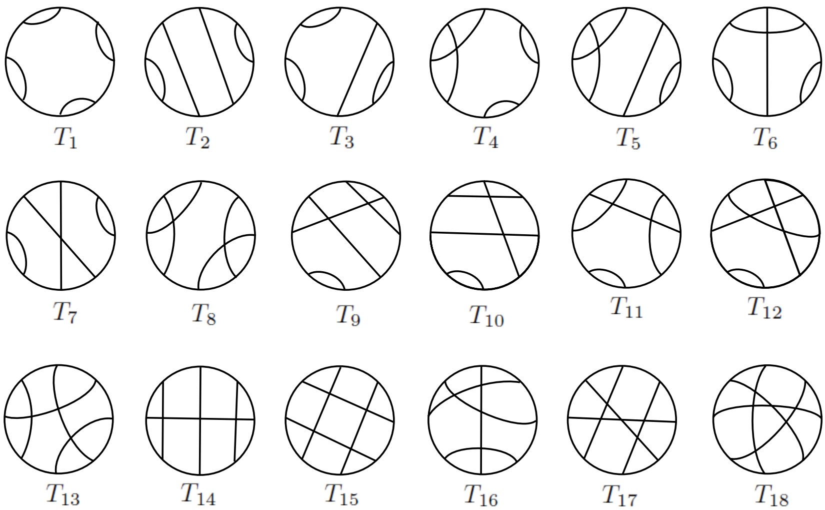


Fig. 14. 4-Chord diagrams.

Each 4-chord-tangle diagram whose shadow is neither $T_{15}$, $T_{17}$ nor $T_{18}$ has an arc $\alpha$ which can be seen as the top or the bottom arc. By removing the arc $\alpha$, we get a 3-chord-tangle diagram. Since the $H(T)$-move is an unknotting operation for any 3-chord-tangle diagram, the $H(T_i)$-move ($i = 1, 2, \ldots, 14$ and 16) is an unknotting operation.

For $T = T_{15}, T_{17}$ and $T_{18}$, if the chord-tangle diagram obtained from $T$ has either the top arc or the bottom arc, the $H(T)$-move is an unknotting operation by the same reason. Two chord-tangle diagrams in Fig. 15 whose shadows are $T_{15}$ have neither the top arc nor the bottom arc. For these two chord-tangle diagrams, the $H(T)$-moves are also unknotting operations because they are the 4(1)-move defined by Ohyama.

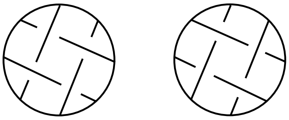

Fig. 15. 4-Chord-tangle diagrams obtained from $T_{15}$.

In Fig. 16, there are eight chord-tangle diagrams whose shadows are $T_{17}$ and these tangle diagrams have neither the top arc nor the bottom arc.

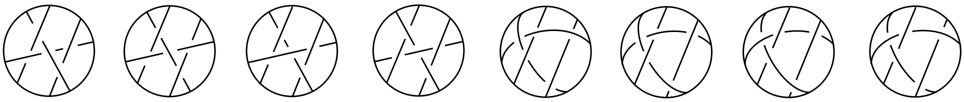

Fig. 16. 4-Chord-tangle diagrams obtained from $T_{17}$.

In Fig. 17, there are 12 chord-tangle diagrams whose shadows are $T_{18}$ and these tangle diagrams have neither the top arc nor the bottom arc.

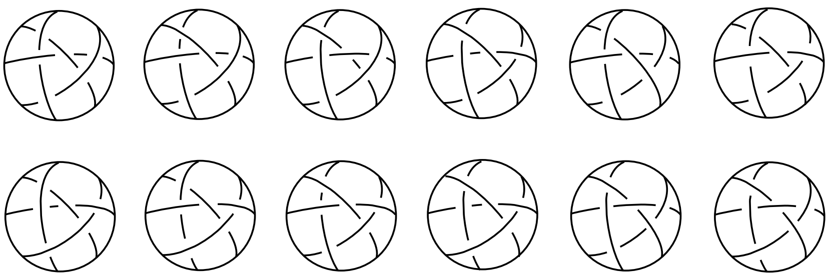

Fig. 17. 4-Chord-tangle diagrams obtained from $T_{18}$.

At the moment, the authors do not know whether the $H(T)$-move is an unknotting operation for 20 diagrams in Figs. 16 and 17.

**Theorem 2.8.** *The $H(T)$-move is an unknotting operation for any* 4*-chord-tangle diagram $T$ except* 20 *diagrams in Figs.* 16 *and* 17*.*

## Acknowledgments

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